Workshop on Multi-body Dynamics

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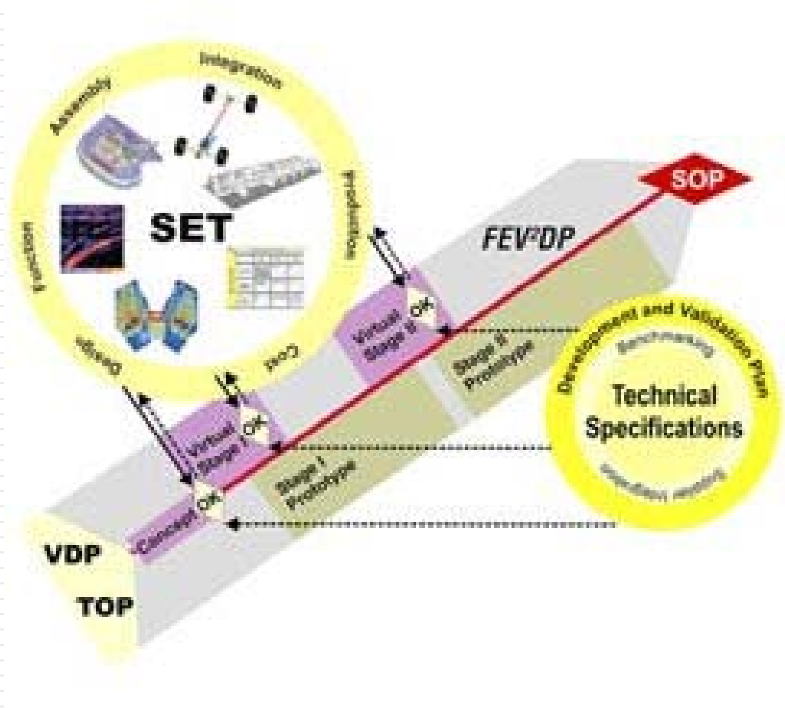
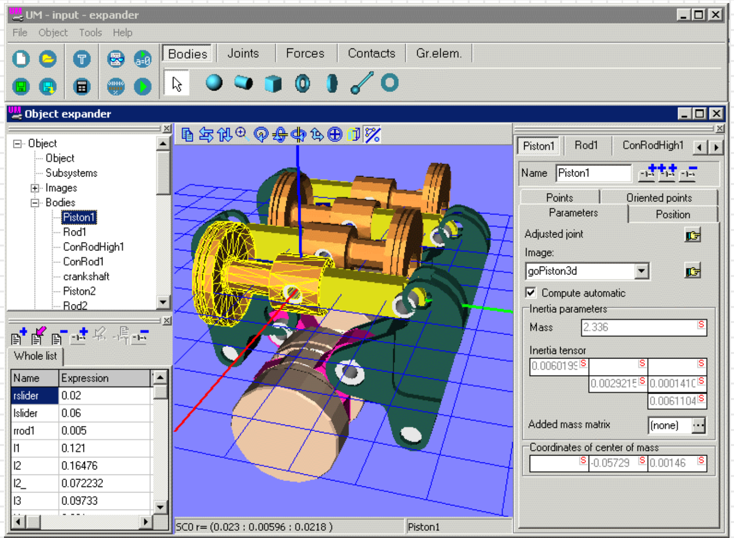
Virtual product development

- Multibody system: mechanical system which is mobile or has mobile parts.
- Multibody dynamics:
  - Computer simulation of the dynamics of multibody systems.
  - It belongs to the general concept of virtual product development.
  - Advantages:
    - Anticipation of system behavior during the first stages of the design process.
    - Reduction of physical prototypes and experimental tests.
    - Consequences: higher quality, lower cost, earlier in the market.



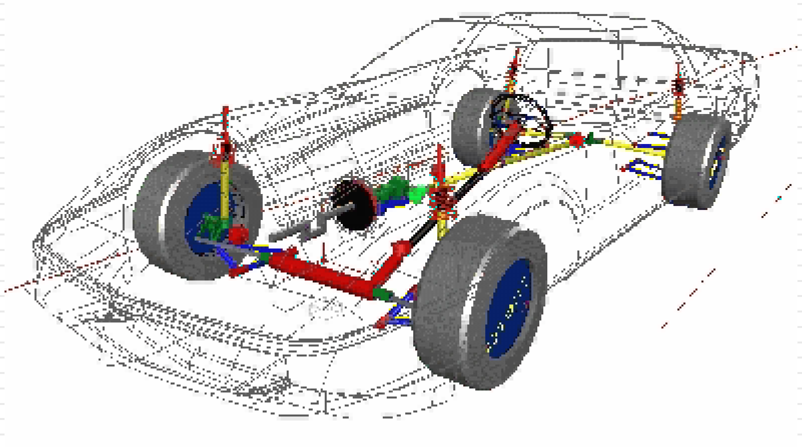
Multibody dynamics

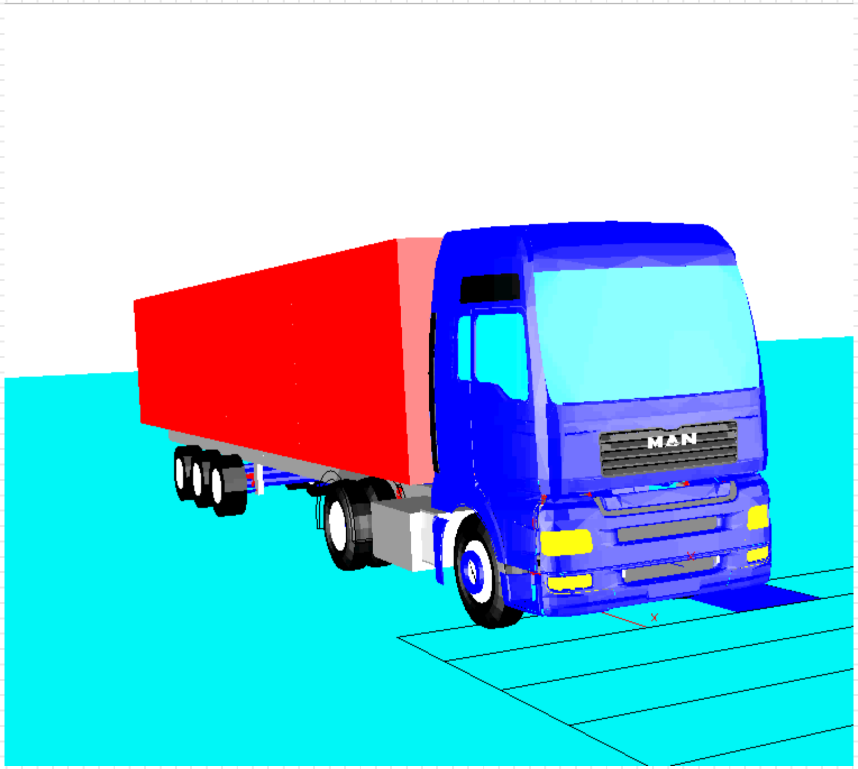
- Computational mechanics of machines and mechanisms: mechanics + numerical methods + programming.
- Enables to solve in the computer the forward dynamics (and the kinematics, and the inverse dynamics) of models of vehicles, machines and mechanisms as detailed as desired.
- Industrial fields of application:
  - Automotive, Aerospatial, Railway, Naval, Energy, Heavy Machinery, Machine-Tool, Robotics, Biomechanics, Health, Sports, Entertaining, etc.



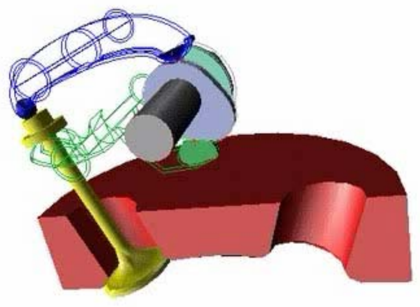
- Application in all the stages of the design process:
  - Design, Simulation, Analysis, Control, Test, Manufacturing and Maintenance.

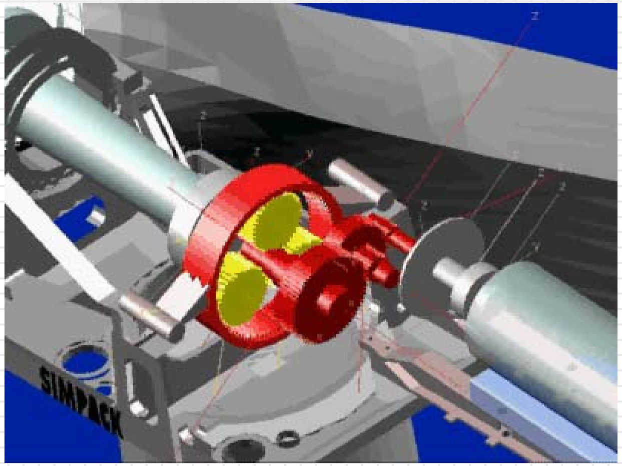
Examples





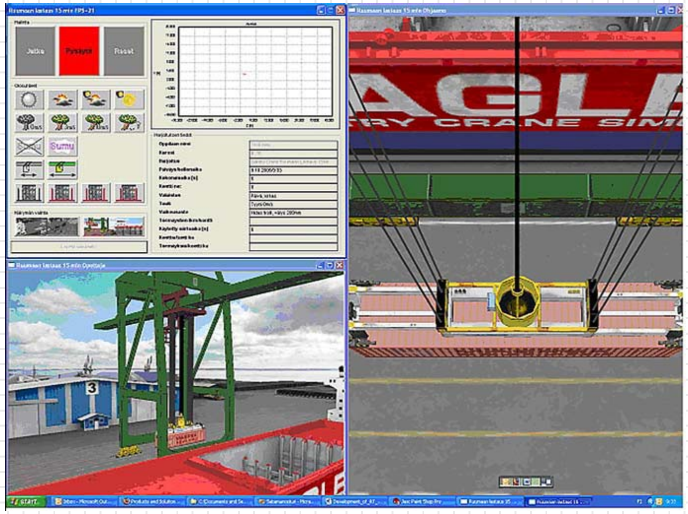
Examples





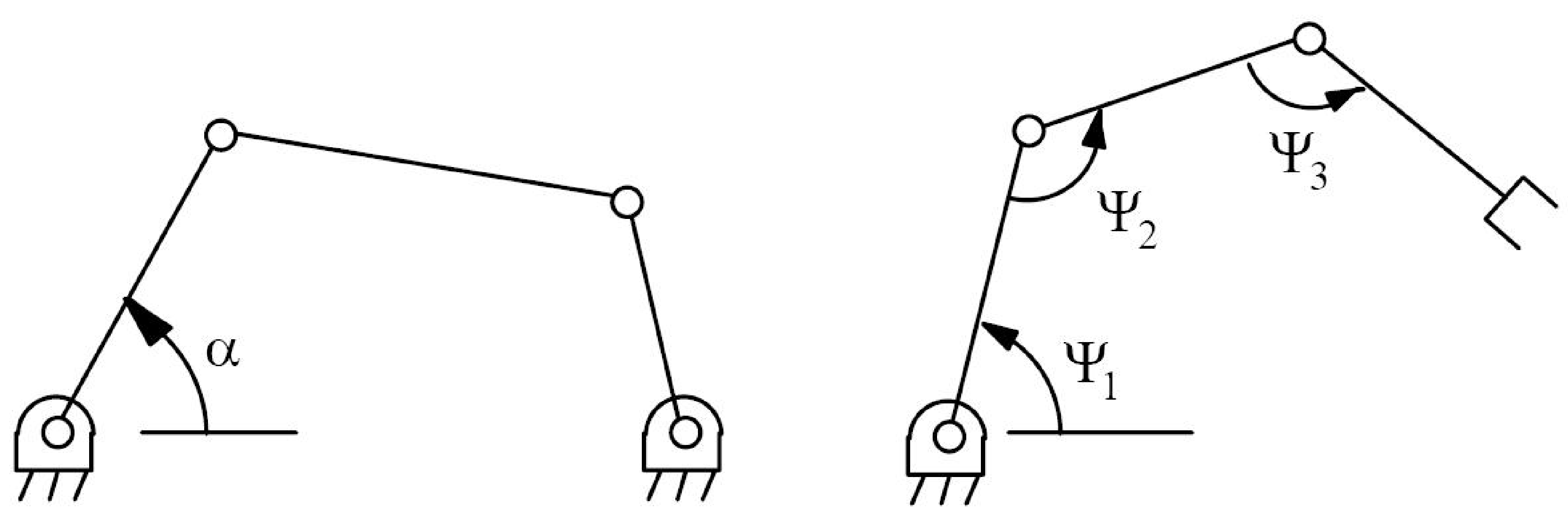
Stages of the multibody dynamics problem

- Problem stages:
  - Physical modeling: simplifications, theories for flexibility, contact, etc.
  - Coordinates selection.
  - Formulation of the equations of motion: kinematics and dynamics.
  - Numerical integration.
  - Implementation: Fortran, C++, Matlab.
- Basic topics:
  - Modeling.
  - Kinematics.
  - Dynamics.



Modeling: traditional method

- Traditionally, mechanisms are modeled in minimum coordinates (Classical Mechanics, Theory of Machines).
- Minimum coordinates: as many as the system degrees of freedom (independent).
- Closed loops are problematic.



Modeling: computational method

- Dependent coordinates: more than the system degrees of freedom.
- Related through constraint equations.

n: number of coordinates

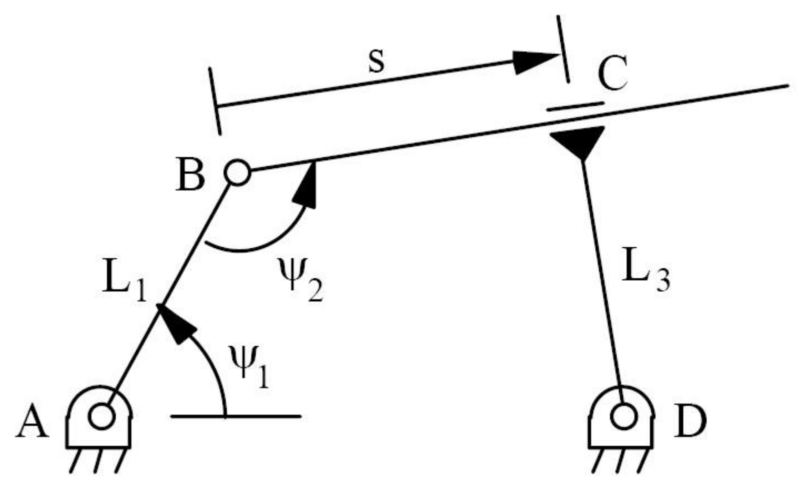
g: number of system degrees of freedom

m: number of constraint equations

$$m=n-g$$

- Three families: relative, reference point, natural.

Modeling: relative coordinates

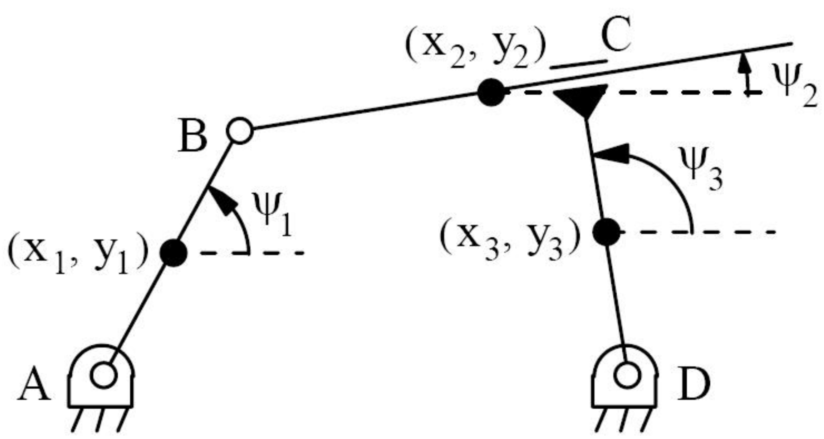


$$\mathbf{AB} + \mathbf{BC} + \mathbf{CD} - \mathbf{AD} = \mathbf{0}$$

$$L_1 \cos\psi_1 + s\cos(\psi_1 + \psi_2 - \pi) + L_3 \operatorname{sen}(\psi_1 + \psi_2 - \pi) - L_4 = 0$$

$$L_1 \operatorname{sen}\psi_1 + s\operatorname{sen}(\psi_1 + \psi_2 - \pi) - L_3 \cos(\psi_1 + \psi_2 - \pi) = 0$$

Modeling: reference point coordinates



$$(x_1 - x_A) - \frac{L_1}{2}\cos\psi_1 = 0$$

$$(y_1 - y_A) - \frac{L_1}{2}\,\mathrm{sen}\,\psi_1 = 0$$

$$(x_1 + \frac{L_1}{2}\cos\psi_1) - (x_2 - \frac{L_2}{2}\cos\psi_2) = 0$$

$$(y_1 + \frac{L_1}{2}\,\mathrm{sen}\,\psi_1) - (y_2 - \frac{L_2}{2}\,\mathrm{sen}\,\psi_2) = 0$$

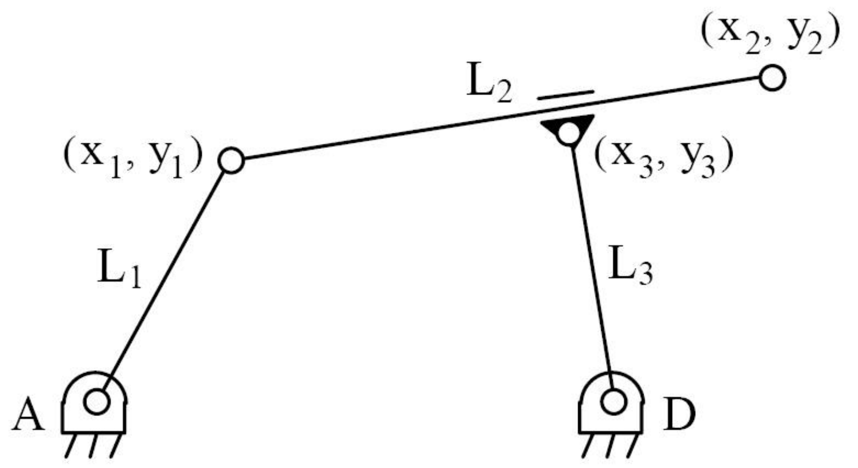
$$\psi_3 - (\psi_2 + \frac{\pi}{2}) = 0$$

$$(x_2 - x_3)\cos\psi_3 + (y_2 - y_3)\,\mathrm{sen}\,\psi_3 - \frac{L_3}{2} = 0$$

$$(x_3 - x_D) - \frac{L_3}{2}\cos\psi_3 = 0$$

$$(y_3 - y_D) - \frac{L_3}{2}\,\mathrm{sen}\,\psi_3 = 0$$

Modeling: natural coordinates



$$(x_1 - x_A)^2 + (y_1 - y_A)^2 - L_1^2 = 0$$

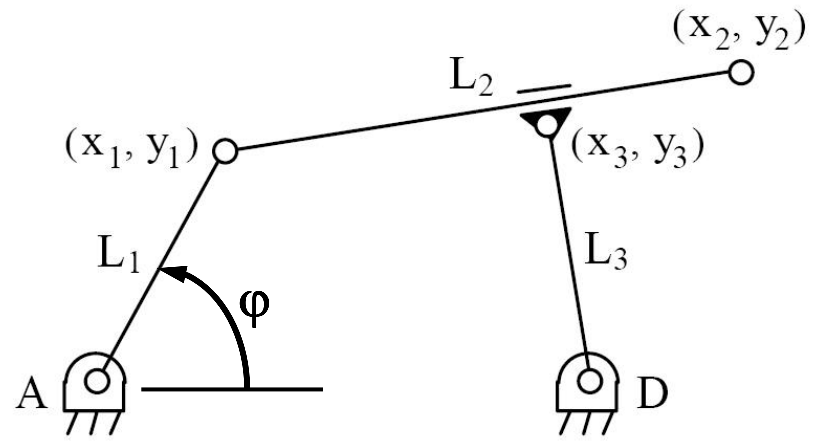
$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - L_2^2 = 0$$

$$(x_3 - x_D)^2 + (y_3 - y_D)^2 - L_3^2 = 0$$

$$(x_2 - x_1)(x_3 - x_D) + (y_2 - y_1)(y_3 - y_D) = 0$$

$$(x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1) = 0$$

Modeling: mixed coordinates (natural + relative)



$$(x_1 - x_A)^2 + (y_1 - y_A)^2 - L_1^2 = 0$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 - L_2^2 = 0$$

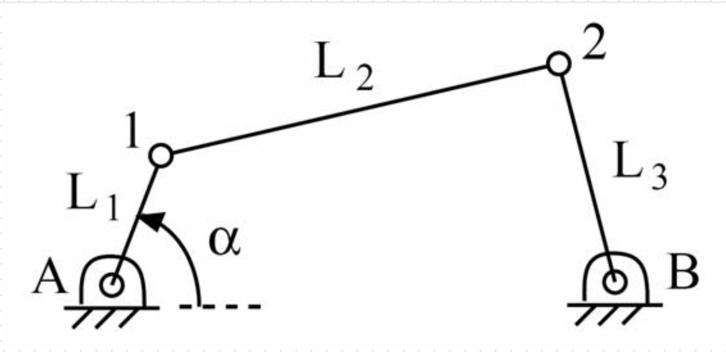
$$(x_3 - x_D)^2 + (y_3 - y_D)^2 - L_3^2 = 0$$

$$(x_2 - x_1)(x_3 - x_D) + (y_2 - y_1)(y_3 - y_D) = 0$$

$$(x_3 - x_1)(y_2 - y_1) - (y_3 - y_1)(x_2 - x_1) = 0$$

$$(x_1 - x_A)(x_D - x_A) + (y_1 - y_A)(y_D - y_A) - L_1 L_4 \cos\varphi = 0$$

Kinematics: position problem

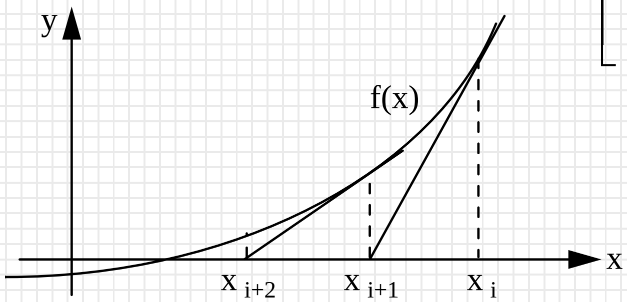


$$\mathbf{q}^t = \{x_1, y_1, x_2, y_2, \alpha\}$$

$$\left.\begin{array}{c}(x_1 - x_A)^2 + (y_1 - y_A)^2 - L_1^2 = 0 \\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - L_2^2 = 0 \\ (x_2 - x_B)^2 + (y_2 - y_B)^2 - L_3^2 = 0 \\ (x_1 - x_A) - L_1 \cos\alpha = 0\end{array}\right\} \mathbf{\Phi}(\mathbf{q}) = 0$$

$$\mathbf{\Phi}(\mathbf{q}) \cong \mathbf{\Phi}(\mathbf{q}_0) + \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_0)(\mathbf{q} - \mathbf{q}_0) = 0 \Longrightarrow \mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_0)(\mathbf{q} - \mathbf{q}_0) = -\mathbf{\Phi}(\mathbf{q}_0)$$

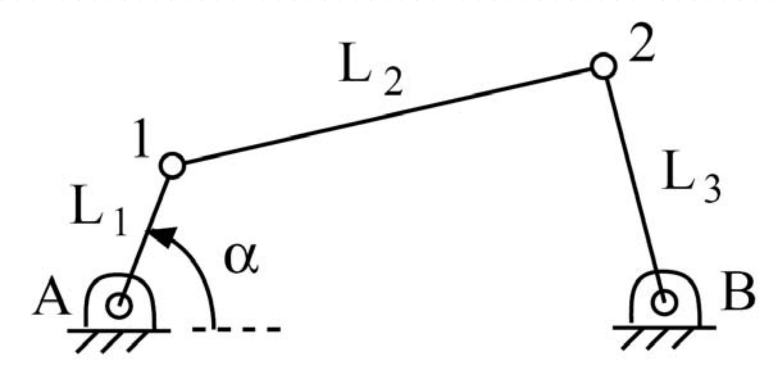
$$\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}) = \begin{bmatrix} 2(x_1 - x_A) & 2(y_1 - x_A) & 0 & 0 & 0 \\ -2(x_2 - x_1) & -2(y_2 - y_1) & 2(x_2 - x_1) & 2(y_2 - y_1) & 0 \\ 0 & 0 & 2(x_2 - x_B) & 2(y_2 - y_B) & 0 \\ 1 & 0 & 0 & 0 & L_1 \operatorname{sen}\alpha \end{bmatrix}$$



$$\boxed{\mathbf{\Phi}_{\mathbf{q}}(\mathbf{q}_i)(\mathbf{q}_{i+1} - \mathbf{q}_i) = -\mathbf{\Phi}(\mathbf{q}_i)}$$

Kinematics: velocity and acceleration problems

$$\mathbf{\Phi}(\mathbf{q}) = \mathbf{0} \quad \Rightarrow \quad \boxed{\mathbf{\Phi_q}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}} \quad \Rightarrow \quad \boxed{\mathbf{\Phi_q}(\mathbf{q})\ddot{\mathbf{q}} = -\dot{\mathbf{\Phi}}_\mathbf{q}\dot{\mathbf{q}}}$$



$$-\dot{\mathbf{\Phi}}_\mathbf{q}\dot{\mathbf{q}} = -\begin{Bmatrix} 2(\dot{x}_1^2 + \dot{y}_1^2) \\ 2[(\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2] \\ 2(\dot{x}_2^2 + \dot{y}_2^2) \\ L_1\dot{\alpha}^2 \cos\alpha \end{Bmatrix}$$

$$\mathbf{\Phi_q}(\mathbf{q}) = \begin{bmatrix} 2(x_1 - x_A) & 2(y_1 - x_A) & 0 & 0 & 0 \\ -2(x_2 - x_1) & -2(y_2 - y_1) & 2(x_2 - x_1) & 2(y_2 - y_1) & 0 \\ 0 & 0 & 2(x_2 - x_B) & 2(y_2 - y_B) & 0 \\ 1 & 0 & 0 & 0 & L_1 \operatorname{sen}\alpha \end{bmatrix}$$

The basic multibody dynamics equations

Lagrange's equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial T}{\partial \mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{t}}\boldsymbol{\lambda} = \mathbf{Q}$$

Kinetic energy

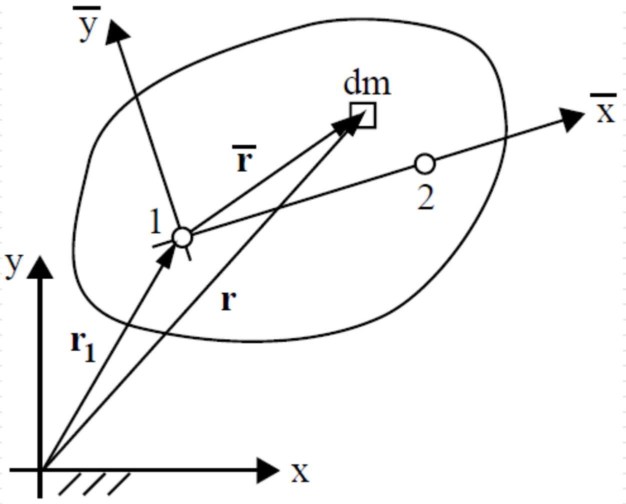
$$T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{t}}\mathbf{M}\dot{\mathbf{q}}$$

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{t}}\boldsymbol{\lambda} = \mathbf{Q}$$

$$\mathbf{\Phi} = \mathbf{0}$$

System of differential-algebraic equations (DAE)

Mass matrix



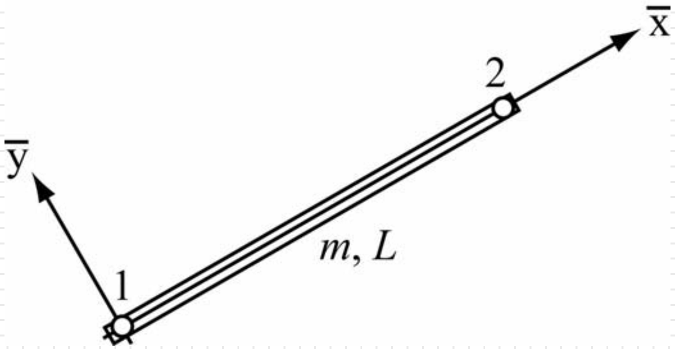
$$\mathbf{r} = \mathbf{r}_1 + \mathbf{A}\bar{\mathbf{r}}$$

$$\mathbf{r}_1 = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} \qquad \mathbf{A} = \frac{1}{L}\begin{pmatrix} x_2 - x_1 & y_1 - y_2 \\ y_2 - y_1 & x_2 - x_1 \end{pmatrix}$$

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_1 + \dot{\mathbf{A}}\bar{\mathbf{r}} \qquad T = \frac{1}{2}\int \dot{\mathbf{r}}^t \dot{\mathbf{r}}\, \mathrm{d}m$$

$$T = \frac{1}{2}\{\dot{x}_1 \dot{y}_1 \dot{x}_2 \dot{y}_2\}\begin{bmatrix} m - 2m\frac{\bar{x}_G}{L} + \frac{I_1}{L^2} & 0 & m\frac{\bar{x}_G}{L} - \frac{I_1}{L^2} & -m\frac{\bar{y}_G}{L} \\ 0 & m - 2m\frac{\bar{x}_G}{L} + \frac{I_1}{L^2} & m\frac{\bar{y}_G}{L} & m\frac{\bar{x}_G}{L} - \frac{I_1}{L^2} \\ m\frac{\bar{x}_G}{L} - \frac{I_1}{L^2} & m\frac{\bar{y}_G}{L} & \frac{I_1}{L^2} & 0 \\ -m\frac{\bar{y}_G}{L} & m\frac{\bar{x}_G}{L} - \frac{I_1}{L^2} & 0 & \frac{I_1}{L^2} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{x}_2 \\ \dot{y}_2 \end{Bmatrix}$$

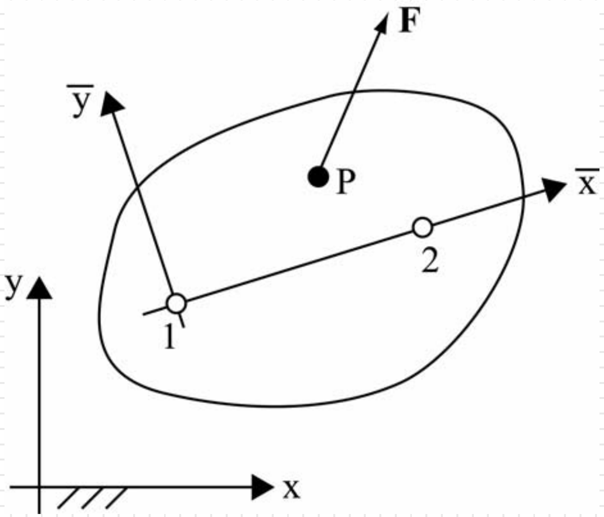
Mass matrix: homogeneous rod



$$\bar{x}_G = \frac{L}{2} \quad ; \quad \bar{y}_G = 0 \quad ; \quad I_1 = \frac{1}{3} m L^2$$

$$\mathbf{M} = \begin{bmatrix} \frac{m}{3} & 0 & \frac{m}{6} & 0 \\ 0 & \frac{m}{3} & 0 & \frac{m}{6} \\ \frac{m}{6} & 0 & \frac{m}{3} & 0 \\ 0 & \frac{m}{6} & 0 & \frac{m}{3} \end{bmatrix}$$

Force vector



$$\mathbf{r}_P = \mathbf{r}_1 + \mathbf{A}\bar{\mathbf{r}}_P$$

$$\mathbf{r}_P = \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} + \frac{1}{L}\begin{pmatrix} x_2 - x_1 & y_1 - y_2 \\ y_2 - y_1 & x_2 - x_1 \end{pmatrix}\begin{Bmatrix} \bar{x}_P \\ \bar{y}_P \end{Bmatrix}$$

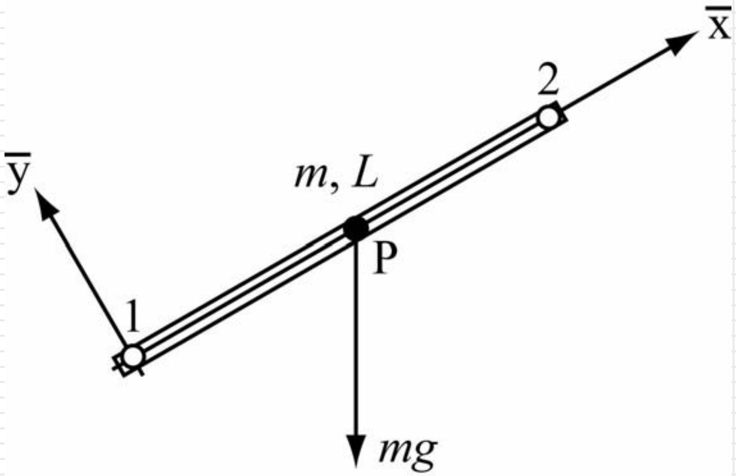
$$\mathbf{r}_P = \frac{1}{L}\begin{bmatrix} L - \bar{x}_P & \bar{y}_P & \bar{x}_P & -\bar{y}_P \\ -\bar{y}_P & L - \bar{x}_P & \bar{y}_P & \bar{x}_P \end{bmatrix}\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{Bmatrix} = \mathbf{C}_P\mathbf{q}$$

$$\dot{\mathbf{r}}_P = \mathbf{C}_P\dot{\mathbf{q}}$$

$$\dot{\tilde{W}} = \mathbf{F}^t\dot{\tilde{\mathbf{r}}}_P = \mathbf{F}^t\mathbf{C}_P\dot{\tilde{\mathbf{q}}} = \mathbf{Q}^t\dot{\tilde{\mathbf{q}}}$$

$$\boxed{\mathbf{Q} = \mathbf{C}_P^T\mathbf{F}}$$

Force vector: weight acting on homogeneous rod

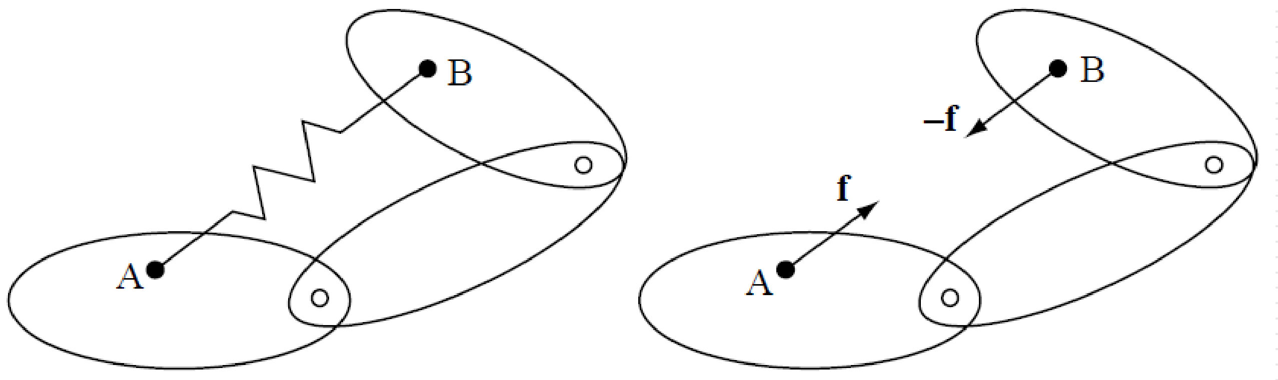


$$\bar{x}_P = \frac{L}{2} \quad ; \quad \bar{y}_P = 0$$

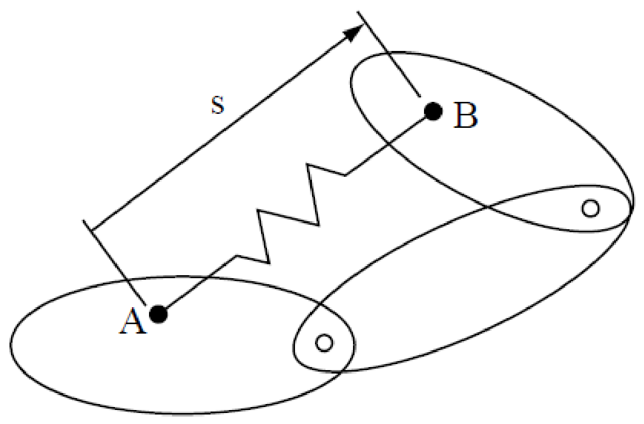
$$\mathbf{C}_P = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \quad ; \quad \mathbf{F} = \begin{Bmatrix} 0 \\ -mg \end{Bmatrix}$$

$$\mathbf{Q} = \begin{Bmatrix} 0 \\ -\frac{mg}{2} \\ 0 \\ -\frac{mg}{2} \end{Bmatrix}$$

Springs, dampers and actuators

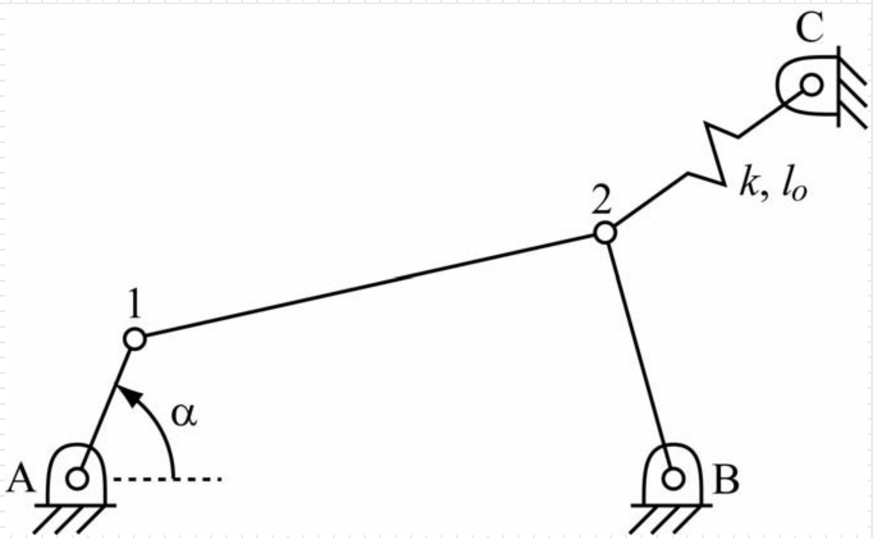


$$\mathbf{f} = -k\left(d_{AB} - l_o\right)\frac{\mathbf{r}_A - \mathbf{r}_B}{d_{AB}}$$



$$f = -k\left(s - l_o\right)$$

Spring acting on four-bar mechanism



$$\mathbf{q} = \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \alpha \end{Bmatrix}$$

$$\mathbf{F} = -k\left(l - l_o\right)\frac{\mathbf{r}_2 - \mathbf{r}_C}{l} \quad ; \quad l = \left|\mathbf{r}_2 - \mathbf{r}_C\right| \quad \Rightarrow \quad \mathbf{Q} = \begin{Bmatrix} 0 \\ 0 \\ F_x \\ F_y \\ 0 \end{Bmatrix}$$

Stabilized Lagrange (Baumgarte)

DAE

$$\boxed{\begin{aligned} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{t}\boldsymbol{\lambda} &= \mathbf{Q} \\ \mathbf{\Phi} &= \mathbf{0} \end{aligned}}$$

$\Rightarrow$ ODE

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{t}\boldsymbol{\lambda} = \mathbf{Q} \\ \ddot{\mathbf{\Phi}} = \mathbf{0} \end{cases} \Rightarrow \begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{T} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q} \\ -\dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} \end{Bmatrix}$$

Unstable

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{t}\boldsymbol{\lambda} = \mathbf{Q} \\ \ddot{\mathbf{\Phi}} + 2\xi\omega\dot{\mathbf{\Phi}} + \omega^{2}\mathbf{\Phi} = \mathbf{0} \end{cases} \Rightarrow$$

Stable

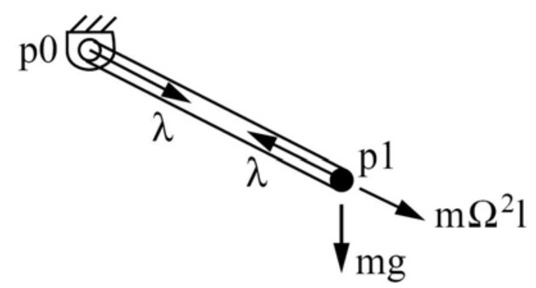
$$\boxed{\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{T} \\ \mathbf{\Phi}_{\mathbf{q}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q} \\ -\dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} - 2\xi\omega\dot{\mathbf{\Phi}} - \omega^{2}\mathbf{\Phi} \end{Bmatrix}}$$

Penalty

DAE

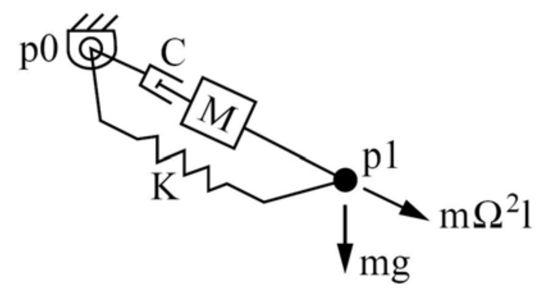
$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{t}\boldsymbol{\lambda} = \mathbf{Q}$$
$$\mathbf{\Phi} = \mathbf{0}$$

⇨ ODE



Aproximation of Lagrange multipliers

$$\boldsymbol{\lambda} = \alpha\left(\ddot{\mathbf{\Phi}} + 2\xi\omega\dot{\mathbf{\Phi}} + \omega^{2}\mathbf{\Phi}\right)$$



$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{T}\alpha\left(\ddot{\mathbf{\Phi}} + 2\xi\omega\dot{\mathbf{\Phi}} + \omega^{2}\mathbf{\Phi}\right) = \mathbf{Q}$$

$$\left(\mathbf{M} + \alpha\mathbf{\Phi}_{\mathbf{q}}^{T}\mathbf{\Phi}_{\mathbf{q}}\right)\ddot{\mathbf{q}} = \mathbf{Q} - \alpha\mathbf{\Phi}_{\mathbf{q}}^{T}\left(\dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} + 2\xi\omega\dot{\mathbf{\Phi}} + \omega^{2}\mathbf{\Phi}\right)$$

Velocity transformation: matrix R

DAE

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{t}\boldsymbol{\lambda} = \mathbf{Q}$$
$$\mathbf{\Phi} = \mathbf{0}$$

⇒ ODE

Transformation to independent coordinates

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{R}\dot{\mathbf{z}} \\ \ddot{\mathbf{q}} = \mathbf{R}\ddot{\mathbf{z}} + \dot{\mathbf{R}}\dot{\mathbf{z}} \end{cases}$$

$$\begin{Bmatrix} \dot{q}_1 \\ \dots \\ \dot{q}_i \\ \dots \\ \dot{q}_n \end{Bmatrix} = \begin{bmatrix} R_{11} & \dots & R_{1i} & \dots & R_{1f} \\ \dots & \dots & \dots & \dots & \dots \\ R_{i1} & \dots & R_{ii} & \dots & R_{if} \\ \dots & \dots & \dots & \dots & \dots \\ R_{n1} & \dots & R_{ni} & \dots & R_{nf} \end{bmatrix} \begin{Bmatrix} 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{Bmatrix}$$

$$\mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R}\ddot{\mathbf{z}} = \mathbf{R}^{\mathrm{T}}\left(\mathbf{Q} - \mathbf{M}\dot{\mathbf{R}}\dot{\mathbf{z}}\right)$$

$$\overline{\mathbf{M}}\ddot{\mathbf{z}} = \overline{\mathbf{Q}}$$

Numerical integrators

- DAE are converted to ODE.
- Many integrators are available for first order ODE of the form:

$$\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$$

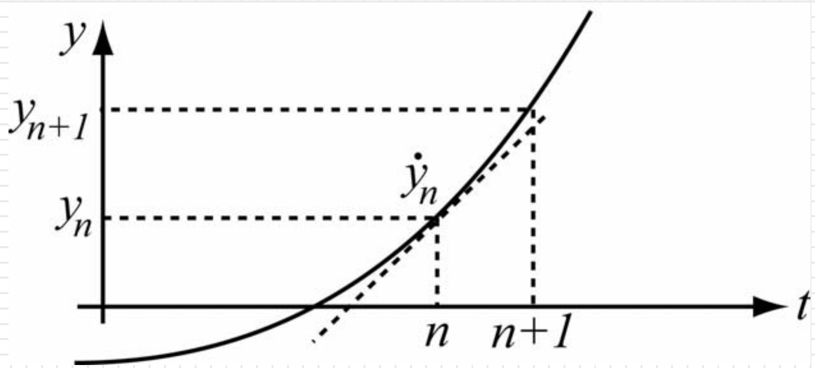
- However, the MBD problem is second order, so variables must be duplicated in order to use such integrators:

$$\mathbf{y} = \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix} \quad \Longrightarrow \quad \dot{\mathbf{y}} = \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix}$$

- Properties of integrators: stability and accuracy.
- *Stiff* problems.

Numerical integrators

- Classification of integrators:
  - Single step vs multistep.
  - Fixed step vs variable step.
  - Explicit vs implicit (iteration: fixed point vs Newton-Raphson).



Forward Euler $\mathbf{y}_{n+1} = \mathbf{y}_n + \dot{\mathbf{y}}_n \Delta t$

Trapezoidal rule $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{\Delta t}{2}\left(\dot{\mathbf{y}}_n + \dot{\mathbf{y}}_{n+1}\right)$

2nd-order explicit Runge-Kutta

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{\Delta t}{2}\left(\dot{\mathbf{y}}_1 + \dot{\mathbf{y}}_2\right)$$

$$\dot{\mathbf{y}}_1 = \mathbf{f}\left(\mathbf{y}_n, t\right)$$

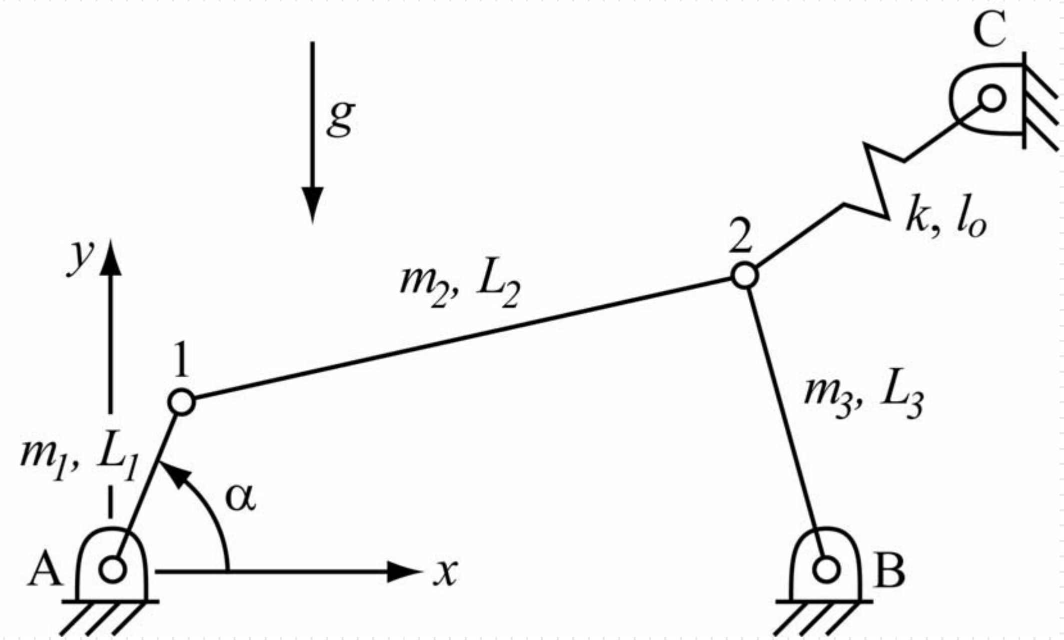
$$\dot{\mathbf{y}}_2 = \mathbf{f}\left(\mathbf{y}_n + \dot{\mathbf{y}}_1 \Delta t, t + \Delta t\right)$$

Adams-Bashforth / Adams-Moulton

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{\Delta t}{24}\left(55\dot{\mathbf{y}}_n - 59\dot{\mathbf{y}}_{n-1} + 37\dot{\mathbf{y}}_{n-2} - 9\dot{\mathbf{y}}_{n-3}\right)$$

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{\Delta t}{24}\left(9\dot{\mathbf{y}}_{n+1} + 19\dot{\mathbf{y}}_n - 5\dot{\mathbf{y}}_{n-1} + \dot{\mathbf{y}}_{n-2}\right)$$

Example: four-bar mechanism



$$\mathbf{q} = \begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \alpha \end{Bmatrix}$$